Discrete Applied Mathematics 158 (2010) 2071–2074

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Small alliances in a weighted graph

Kenji Kimura [a], Masayuki Koyama [b], Akira Saito [b,*]

[a] *Department of Management Science, Tokyo University of Science, Japan*

[b] *Department of Computer Science, Nihon University, Sakurajosui 3–25–40, Setagaya-Ku, Tokyo 156–8550, Japan*

ARTICLE INFO

Article history:
Received 16 February 2010
Received in revised form 30 August 2010
Accepted 6 September 2010
Available online 25 September 2010

Keywords:
Defensive alliance
Alliance
Weighted graph

ABSTRACT

We extend the notion of a defensive alliance to weighted graphs. Let (G, w) be a weighted graph, where G is a graph and w be a function from $V(G)$ to the set of positive real numbers. A non-empty set of vertices S in G is said to be a weighted defensive alliance if $\sum_{x\in N_G(v)\cap S} w(x) + w(v) \geq \sum_{x\in N_G(v)-S} w(x)$ holds for every vertex v in S. Fricke et al. (2003) [3] have proved that every graph of order n has a defensive alliance of order at most $\lceil \frac{1}{2}n \rceil$. In this note, we generalize this result to weighted defensive alliances. Let G be a graph of order n. Then we prove that for any weight function w on $V(G)$, (G, w) has a defensive weighted alliance of order at most $\lceil \frac{1}{2}n \rceil$. We also extend the notion of strong defensive alliance to weighted graphs and generalize a result in Fricke et al. (2003) [3].

1. Introduction

A non-empty set of vertices S in a graph G is said to be a *defensive alliance*, or simply an *alliance*, if

$$|N_G(v) \cap S| + 1 \geq |N_G(v) - S| \tag{1}$$

holds for every $v \in S$, where $N_G(v)$ denotes the neighborhood of v in G. The study of alliances in graphs was initiated by Kristiansen et al. [5]. It is a mathematical model of allies among countries during a time of war. In this model, a vertex represents a country, and each country can send forces to neighboring countries to defend them. By the definition of the alliance, no country in the alliance is outnumbered. Thus, the countries in the alliance are defended, if each country has the same amount of forces.

In the real world, however, countries have different amount of forces. One strong country may smash a country in an alliance formed by a number of weak countries. If we consider such a situation in the model, it is more natural to consider an alliance in a weighted graph. For a graph G and a function $w: V(G) \to \boldsymbol{R}^+$, we call the pair (G, w) a weighted graph. A non-empty set of vertices S in G is said to be a *weighted defensive alliance*, or simply a *weighted alliance*, if

$$\sum_{x\in N_G(v)\cap S} w(x) + w(v) \geq \sum_{x\in N_G(v)-S} w(x)$$

holds for every $x \in S$. If w is a constant function taking value one, then a weighted alliance coincides with an ordinary alliance in an unweighted graph.

In this note, we study the order of a smallest weighted alliance. For unweighted graphs, in [5], it was conjectured that a graph of order n has an alliance of order at most $\lceil \frac{1}{2}n \rceil$. This conjecture was proved by Fricke et al. [3].

Theorem A (*[3]*). *Every graph of order n has an alliance of order at most $\lceil \frac{1}{2}n \rceil$.*

* Corresponding author. Fax: +81 3 5317 9770.
E-mail addresses: kimura@ms.kagu.tus.ac.jp (K. Kimura), asaito@chs.nihon-u.ac.jp (A. Saito).

0166-218X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2010.09.001

We generalize this result to weighted alliances. Let G be a graph of order n. Then we prove that for any weight function w on $V(G)$, the weighted graph (G, w) has a weighted alliance of order at most $\lceil \frac{1}{2}n \rceil$. Note that this bound depends only on the order of G and does not depend on the weight function w.

We are not the first one to introduce the notion of a weight alliance. In [4], Jamieson and Dean introduced it and discussed its algorithmic aspects.

In the next section, we give our main theorem and its proof. In Section 3, we will discuss the extension of a strong alliance to weighted graphs. And in Section 4, we give concluding remarks.

For standard graph-theoretic terminology, we refer the reader to [1]. A pair of disjoint subsets A and B of a set X is said to be a partition of X if neither A nor B is an empty set and $X = A \cup B$.

2. Small alliances in a weighted graph

In this section, we prove the following theorem.

Theorem 1. *Every weighted graph of order n has a weighted alliance of order at most $\lceil \frac{1}{2}n \rceil$.*

In [5], Kristiansen et al. remarked that a smallest alliance of the complete graph of order n has order $\lceil \frac{1}{2}n \rceil$. It gives the sharpness of Theorem 1.

In order to prove Theorem 1, we introduce one parameter. Let (G, w) be a weighted graph and let $\{X, Y\}$ be a partition of $V(G)$. Then define $s(X, Y)$ by

$$s(X, Y) = \sum_{\substack{xy \in E(G) \\ x \in X, y \in Y}} w(x)w(y).$$

Note that in the case of unweighted graphs, w is a constant function taking value one and $s(X, Y)$ is simply the number of edges between X and Y.

The next lemma is a simple but useful observation for the subsequent arguments.

Lemma 2. *Let (G, w) be a weighted graph and $\{X, Y\}$ a partition of $V(G)$. Then for $u \in X$,*

$$s(X - \{u\}, Y \cup \{u\}) = s(X, Y) + w(u) \left(\sum_{x \in N_G(u) \cap X} w(x) - \sum_{y \in N_G(u) \cap Y} w(y) \right).$$

Proof. The proof is a simple calculation.

$$\begin{aligned} s(X - \{u\}, Y \cup \{u\}) &= \sum_{\substack{xy \in E(G) \\ x \in X, y \in Y}} w(x)w(y) + \sum_{\substack{xu \in E(G) \\ x \in X}} w(x)w(u) - \sum_{\substack{uy \in E(G) \\ y \in Y}} w(u)w(y) \\ &= s(X, Y) + w(u) \left(\sum_{\substack{xu \in E(G) \\ x \in X}} w(x) - \sum_{\substack{uy \in E(G) \\ y \in Y}} w(y) \right) \\ &= s(X, Y) + w(u) \left(\sum_{x \in N_G(u) \cap X} w(x) - \sum_{y \in N_G(u) \cap Y} w(y) \right). \quad \square \end{aligned}$$

Now we prove Theorem 1.

Proof of Theorem 1. Let (G, w) be a weighted graph of order n. Since $V(G)$ is a weighted alliance, the theorem holds if $n = 1$. Hence we may assume $n \geq 2$. Choose $X \subset V(G)$ with $|X| = \lfloor \frac{1}{2}n \rfloor$ so that $s(X, V(G) - X)$ is as small as possible. Let $Y = V(G) - X$. Then $|Y| = \lceil \frac{1}{2}n \rceil$. Note in particular that since $n \geq 2$, neither X nor Y is an empty set and hence $\{X, Y\}$ is a partition.

We claim that either X or Y is a weighted alliance, which yields the theorem. Assume, to the contrary, that neither X nor Y is a weighted alliance. Then for some $u \in X$ and $v \in Y$, we have

$$\sum_{x \in N_G(u) \cap X} w(x) + w(u) < \sum_{y \in N_G(u) \cap Y} w(y)$$

and

$$\sum_{y \in N_G(v) \cap Y} w(y) + w(v) < \sum_{x \in N_G(v) \cap X} w(x).$$

Let $X' = X - \{u\}$, $Y' = Y \cup \{u\}$, $X'' = X' \cup \{v\}$ and $Y'' = Y' - \{v\}$. Then $|X''| = |X| = \lfloor \frac{1}{2}n \rfloor$ and $|Y''| = |Y| = \lceil \frac{1}{2}n \rceil$. By Lemma 2, we have

$$\begin{aligned} s(X', Y') &= s(X, Y) + w(u) \left(\sum_{x \in N_G(u) \cap X} w(x) - \sum_{y \in N_G(u) \cap Y} w(y) \right) \\ &< s(X, Y) - w(u)^2. \end{aligned}$$

Lemma 2 also yields

$$s(X'', Y'') = s(X', Y') + w(v) \left(\sum_{y \in N_G(v) \cap Y'} w(y) - \sum_{x \in N_G(v) \cap X'} w(x) \right).$$

If $uv \notin E(G)$, then $N_G(v) \cap X' = N_G(v) \cap X$ and $N_G(v) \cap Y' = N_G(v) \cap Y$. Thus

$$\begin{aligned} s(X'', Y'') &= s(X', Y') + w(v) \left(\sum_{y \in N_G(v) \cap Y} w(y) - \sum_{x \in N_G(v) \cap X} w(x) \right) \\ &< s(X', Y') - w(v)^2 < s(X, Y) - w(u)^2 - w(v)^2 < s(X, Y). \end{aligned}$$

If $uv \in E(G)$, then

$$\begin{aligned} \sum_{y \in N_G(v) \cap Y'} w(y) - \sum_{x \in N_G(v) \cap X'} w(x) &= \sum_{y \in N_G(v) \cap Y} w(y) + w(u) - \left(\sum_{x \in N_G(v) \cap X} w(x) - w(u) \right) \\ &= \sum_{y \in N_G(v) \cap Y} w(y) - \sum_{x \in N_G(v) \cap X} w(x) + 2w(u) \\ &< -w(v) + 2w(u). \end{aligned}$$

Therefore,

$$\begin{aligned} s(X'', Y'') &< s(X', Y') + w(v)\big(-w(v) + 2w(u)\big) \\ &< s(X, Y) - w(u)^2 - w(v)^2 + 2w(u)w(v) \\ &= s(X, Y) - \big(w(u) - w(v)\big)^2 \le s(X, Y). \end{aligned}$$

Thus, we have $s(X'', Y'') < s(X, Y)$ in either case. This contradicts the minimality of $s(X, Y)$, and the theorem follows. □

3. Strong alliances

A non-empty set of vertices S in a graph is said to be a strong alliance if $|N_G(v) \cap S| \ge |N_G(v) - S|$ holds for every $v \in S$. Comparing this definition with (1) in the definition of an alliance, we can interpret the meaning of "strong" in two ways.

(1) The contribution of $+1$ from v is not counted.
(2) The contribution of $+1$ from v is counted, but the inequality is strict.

These two interpretations do not make any difference when we consider unweighted graphs. But they do make a difference for weighted graphs.

In this note, we adopt the first interpretation, which leads us to the following definition. Let (G, w) be a weighted graph. Then a non-empty set of vertices S in G is said to be a *strong weighted alliance* if

$$\sum_{x \in N_G(v) \cap S} w(x) \ge \sum_{x \in N_G(v) - S} w(x)$$

holds for every $v \in S$.

To make a comparison, we also give a definition to the second interpretation. A non-empty set of vertices S in a weighted graph (G, w) is said to be a *strict weighted alliance* if

$$\sum_{x \in N_G(v) \cap S} w(x) + w(v) > \sum_{x \in N_G(v) - S} w(x)$$

holds for every $v \in S$. It is obvious that every strong weighted alliance is a strict weighted alliance. Therefore, an upper bound on the order of a smallest strong weighted alliance also applies to the order of a smallest strict weighted alliance.

In [3], Fricke et al. have proved that every graph of order n has a strong alliance of order at most $\lfloor \frac{1}{2}n \rfloor + 1$. We give the same upper bound for weighted graphs.

Theorem 3. *Every weighted graph of order n has a strong weighted alliance of order at most* $\left\lfloor \frac{1}{2}n \right\rfloor + 1$.

Note that the conclusion does not depend on the weight function.

Proof of Theorem 3. Let (G, w) be a weighted graph of order n. Since $V(G)$ is a strong alliance of G, the theorem holds if $n \leq 2$. Hence we may assume $n \geq 3$. Take a partition $\{X, Y\}$ of $V(G)$ with $||X| - |Y|| \leq 2$ so that $s(X, Y)$ is as small as possible. Then $\max\{|X|, |Y|\} \leq \left\lfloor \frac{1}{2}n \right\rfloor + 1$, and hence it suffices to prove that either X or Y is a strong weighted alliance.

Assume, to the contrary, that neither X nor Y is a strong alliance. By symmetry, we may assume $|X| \geq |Y|$. Since X is not a strong weighted alliance, $\sum_{x \in N_G(u) \cap X} w(x) < \sum_{y \in N_G(u) \cap Y} w(y)$ for some $u \in X$. Let $X' = X - \{u\}$ and $Y' = Y \cup \{u\}$. Since $n \geq 3$, $|X| \geq 2$ and hence $X' \neq \emptyset$. Therefore, $\{X', Y'\}$ is a partition of $V(G)$. Moreover, since $0 \leq |X| - |Y| \leq 2$, we have $||X'| - |Y'|| \leq 2$. On the other hand, by Lemma 2, we have

$$s(X', Y') = s(X, Y) + w(u) \left(\sum_{x \in N_G(u) \cap X} w(x) - \sum_{y \in N_G(u) \cap Y} w(y) \right) < s(X', Y').$$

This contradicts the minimality of $s(X, Y)$, and the theorem follows. □

4. Concluding remarks

We have extended the notions of an alliance and a strong alliance to weighted graphs. And we have proved that a weighted graph of order n has a weighted alliance of order at most $\left\lceil \frac{1}{2}n \right\rceil$ and a strong weighted alliance of order at most $\left\lfloor \frac{1}{2}n \right\rfloor + 1$.

The proof strategies in this note are the same as those in [3]. The difference is the introduction of a new parameter $s(X, Y)$. It enables us to extend the arguments in [3] to weighted graphs. For unweighted graphs, it coincides with the number of edges between X and Y.

As a notion similar to an alliance, Kristiansen et al. [5] also introduced an offensive alliance. For a non-empty set of vertices S in a graph G, define ∂S by $\partial S = N_G(S) - S = \{v \in V(G) - S : N_G(v) \cap S \neq \emptyset\}$. Then S is said to be an offensive alliance if $|N_G(v) \cap S| \geq |N_G(v) - S| + 1$ holds for every $v \in \partial S$. We can extend this definition to weighted graph just as we have done for a defensive alliance. A set of vertices S in a weighted graph (G, w) is said to be a *weighted offensive alliance* if $\sum_{x \in N_G(v) \cap S} w(x) \geq \sum_{y \in N_G(v) - S} w(y) + w(v)$ holds for every $v \in \partial S$. Favaron et al. [2] proved that every graph of order n has an offensive alliance of order at most $\left\lfloor \frac{2}{3}n \right\rfloor$. We tried to extend their arguments with $s(X, Y)$, but we could not obtain an upper bound on the order of a smallest weighted offensive alliance. Therefore, we pose one problem at the end of this note.

Problem. Give an upper bound to the order of a smallest weighted offensive alliance in a weighted graph of order n.

Acknowledgement

The third author was partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), 19500017, 2009.

References

[1] G. Chartrand, L. Lesniak, Graphs & Digraphs, 4th ed., Chapman and Hall/CRC, Boca Raton, FL, USA, 2005.
[2] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar, R.D. Skaggs, Offensive alliances in graphs, Discuss. Math. Graph Theory 24 (2004) 263–275.
[3] G.H. Fricke, L.M. Lawson, T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, A note on defensive alliances in graphs, Bull. Inst. Combin. Appl. 38 (2003) 37–41.
[4] L.H. Jamieson, B.C. Dean, Weighted alliances in graphs, Congr. Numer. 187 (2007) 76–82.
[5] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004) 157–177.